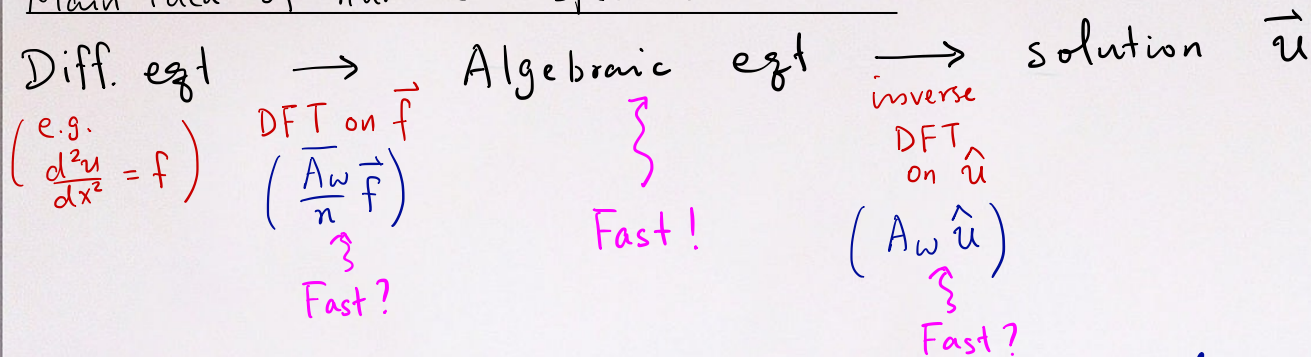


## Lecture 11:

### Main idea of numerical spectral method



Remark: To develop an efficient numerical spectral method, we need to compute  $A_w \hat{u}$  and  $\frac{\overline{A_w} \vec{f}}{n}$  fast.

- Computational cost for  $A_w \hat{u}$  is  $\mathcal{O}(n^2)$ .  
( $n \times n$ )

Goal: Reduce the computational cost to  $\mathcal{O}(n \log n)$

e.g.  $n = 2^{10}$ ,  $n^2 = 2^{20}$ ,  $n \log n = 10 \cdot 2^{10} < 2^{14}$ .  $\therefore 2^6 = 64$  times faster!

# Fast Fourier Transform (FFT) (Cooley and Tukey, 1965)

Let  $F_n = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & \omega_n & \omega_n^2 & \dots & \omega_n^{n-1} \\ \vdots & \vdots & & & \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \dots & \omega_n^{(n-1)^2} \end{pmatrix} \in M_{n \times n}$  where  $\omega_n = e^{i(\frac{2\pi}{n})}$

Let  $\vec{y} = F_n \vec{X}$ , where  $\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix}$  and  $\vec{X} = \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_{n-1} \end{pmatrix}$ . Suppose  $n = 2m$ .

Then: for each  $0 \leq j \leq n-1$ ,

$$y_j = \sum_{k=0}^{n-1} \omega_n^{jk} X_k = \sum_{k=0}^{2m-1} \omega_{2m}^{kj} X_k$$

Divide  $k=0, 1, 2, \dots, 2m-1$  into two parts:

Part 1:  $0, 2, 4, 6, \dots, 2(m-1)$  (Even)

Part 2:  $1, 3, 5, 7, \dots, 2m-1$  (Odd)

$$y_j = \underbrace{\sum_{k=0}^{m-1} \omega_{2m}^{2kj} X_{2k}}_{\text{Part 1}} + \underbrace{\sum_{k=0}^{m-1} \omega_n^{(2k+1)j} X_{2k+1}}_{\text{Part 2}}$$

$$= \sum_{k=0}^{m-1} \omega_m^{kj} X_{2k} + \sum_{k=0}^{m-1} \omega_n^j \omega_m^{kj} X_{2k+1}$$

$\left( \because \omega_{2m}^{2k} = e^{i \left( \frac{2\pi}{2m} \right) 2k} \right)$   
 $= \omega_m^k$

Denote  $\vec{X}' = \begin{pmatrix} X_0 \\ X_2 \\ \vdots \\ X_{2m-2} \end{pmatrix}$ ,  $\vec{X}'' = \begin{pmatrix} X_1 \\ X_3 \\ \vdots \\ X_{2m-1} \end{pmatrix}$ . Denote  $\vec{y}' = \underset{\substack{\uparrow \\ M \times m}}{F_m} \vec{X}'$  and  $\vec{y}'' = F_m \vec{X}''$ .

$$\therefore y_j = \underbrace{\left( F_m \vec{X}' \right)_j}_{\frac{1}{2} \times \frac{1}{2}} + \omega_n^j \underbrace{\left( F_m \vec{X}'' \right)_j}_{\frac{1}{2} \times \frac{1}{2}} \text{ for } j=0, 1, 2, \dots, m-1$$

$$= \left( \vec{y}' \right)_j + \omega_n^j \left( \vec{y}'' \right)_j \quad j = m, m+1, \dots, 2m-1$$

$$y_{j+m} = \sum_{k=0}^{m-1} \omega_n^{2k(j+m)} x_{2k} + \sum_{k=0}^{m-1} \omega_n^{(2k+1)(j+m)} x_{2k+1} \quad \text{for } j=0, 1, 2, \dots, m-1$$

Capture  $y_m, y_{m+1}, \dots, y_{2m-1}$

$$= \sum_{k=0}^{m-1} \omega_m^{kj} \omega_m^{2km} (\vec{x}')_k + \sum_{k=0}^{m-1} \omega_m^{k(j+m)} \omega_n^{j+m} (\vec{x}'')_k$$

$e^{i(\frac{2\pi}{m})k m}$  " 1  
 $\omega_m^{kj}$  "  $\omega_m^{km}$   
 $\omega_n^{j+m}$  "  $\omega_n^m$   
 $e^{i(\frac{2\pi}{2m}) \cdot m}$  " 1

$$\therefore y_{j+m} = \sum_{k=0}^{m-1} \omega_m^{kj} (\vec{x}')_k + \sum_{k=0}^{m-1} (-\omega_n^j) \omega_m^{kj} (\vec{x}'')_k$$

$$y_{j+m} = (F_m \vec{x}')_j - \omega_n^j (F_m \vec{x}'')_j \quad \text{for } j=0, 1, 2, \dots, m-1$$

Note:  $n \times n$  matrix multiplication becomes  $\frac{n}{2} \times \frac{n}{2} = m \times m$  matrix multiplication.

Denote: 
$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \otimes \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} v_1 w_1 \\ v_2 w_2 \\ \vdots \\ v_n w_n \end{pmatrix}$$

Then: 
$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{m-1} \end{pmatrix} = \underbrace{\vec{y}'}_{F_m \vec{x}'} + \begin{pmatrix} W_n^0 \\ W_n^1 \\ W_n^2 \\ \vdots \\ W_n^{m-1} \end{pmatrix} \otimes \underbrace{\vec{y}''}_{F_m \vec{x}''} \quad \text{and} \quad \begin{pmatrix} y_m \\ y_{m+1} \\ y_{m+2} \\ \vdots \\ y_{2m-1} \end{pmatrix} = \vec{y}' - \begin{pmatrix} W_n^0 \\ W_n^1 \\ W_n^2 \\ \vdots \\ W_n^{m-1} \end{pmatrix} \otimes \vec{y}''$$

Summary of FFT ( $n=2m$ )

Step 1: Split  $\vec{x}$  into  $\vec{x}' = \begin{pmatrix} x_0 \\ x_2 \\ \vdots \\ x_{2m-2} \end{pmatrix}$ ,  $\vec{x}'' = \begin{pmatrix} x_1 \\ x_3 \\ \vdots \\ x_{2m-1} \end{pmatrix}$

Step 2: Compute  $\vec{y}' = F_m \vec{x}'$  and  $\vec{y}'' = F_m \vec{x}''$

Step 3: Compute  $F_{\frac{m}{2}} \rightsquigarrow F_{\frac{m}{4}} \dots$

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_{m-1} \end{pmatrix} = \underbrace{\vec{y}'}_{F_m \vec{x}'} + \begin{pmatrix} W_n^0 \\ W_n^1 \\ W_n^2 \\ \vdots \\ W_n^{m-1} \end{pmatrix} \otimes \underbrace{\vec{y}''}_{F_m \vec{x}''} \quad \text{and} \quad \begin{pmatrix} y_m \\ y_{m+1} \\ y_{m+2} \\ \vdots \\ y_{2m-1} \end{pmatrix} = \vec{y}' - \begin{pmatrix} W_n^0 \\ W_n^1 \\ W_n^2 \\ \vdots \\ W_n^{m-1} \end{pmatrix} \otimes \vec{y}''$$

